Calculus 140, section 5.3 Special Properties of the Definite Integral
7 tidbits about integrals you didn't know you needed to know
notes by Tim Pilachowski

Definition 5.6, part 1: "Let f be continuous on [a, b]. Then $\int_a^a f(x)\,dx = 0$."

It's fairly obvious why this must be true.

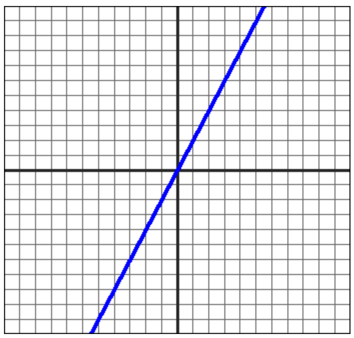
$$\int_a^a f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{1} f(t_k)\,\Delta x_k = \lim_{\|P\|\to 0} f(t_k) * 0 = 0$$

In geometric area terms, a line segment, which is infinitely thin, has a width, and therefore an area, equal to 0. Among other things this means that the areas under a curve on the intervals [a, b], [a, b), (a, b] and (a, b) are all mathematically equal.

Definition 5.6, part 2: "Let f be continuous on [a, b]. Then $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$."

We've already encountered a similar concept when we looked at distance and velocity: positive is up or forward, and negative is down or backward. The "integral from b to a" is the "integral from a to b" in reverse gear.

5.2 Example B revisited: Evaluate $\int_2^2 2x\,dx$ and $\int_5^2 2x\,dx$.



Theorem 5.7, Rectangle Property: "For any numbers a, b, and c, $\int_a^b c\,dx = c(b-a)$."

Proof:

5.2 Example A revisited: Evaluate $\int_{10}^2 5\,dx$.

Theorem 5.8, Addition Property: "Let f be continuous on an interval containing a, b, and c. Then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.\text{"}$$

The text has put the highly-technical proof in the Appendix.

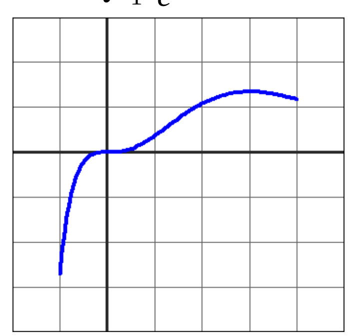
Example C: Given the piecewise continuous function $f(x) = \begin{cases} 2x & x < 3 \\ 5 & x \geq 3 \end{cases}$, evaluate $\int_0^6 f(x)\,dx$.

"Piecewise continuous" means the function is composed of continuous pieces.
(See the text for the detailed explanation.)

Theorem 5.9, Comparison Property: "Let f be continuous on [a, b], and suppose $m \leq f(x) \leq M$ for all x in [a, b]. Then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$."

The value $m(b-a)$ is a **lower bound** for the integral; $M(b-a)$ is an **upper bound** for the integral.
The text's proof, based on a Riemann sum, takes 5 lines of text and 3 lines of equations.

4.1 Example C revisited: Using the Comparison Property, find lower and upper bounds for $\int_{-1}^4 \frac{x^3}{e^x}\,dx$.



Corollary 5.10: "Let f be nonnegative and continuous on [a, b]. Then $\int_a^b f(x)\,dx \geq 0$."

Proof:

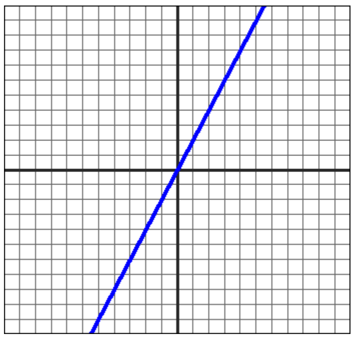
Now we come to the 7th and final tidbit of this section.

Theorem 5.11, Mean Value Theorem for Integrals: "Let f be continuous on [a, b]. Then there is a number c in [a, b] such that $\int_a^b f(x)\,dx = f(c)(b-a)$."

Proof:

The value $\frac{\int_b^a f(x)\,dx}{b-a} = \frac{1}{b-a}\int_b^a f(x)\,dx$ is called the **mean value** or **average value** of f on [a, b].

5.2 Example B once again: Find the mean value of $f(x) = 2x$ on [2, 5].



interpretation in calculus terms:

interpretation in geometric terms: